Algorithmic development of tensor networks for HEP
&
Ultra-cold Atom Simulator for Quantum Link Model's in (2+1)D

Peter Majcen

University of Padua, INFN Padua

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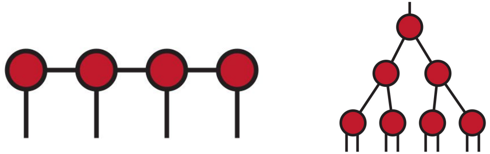
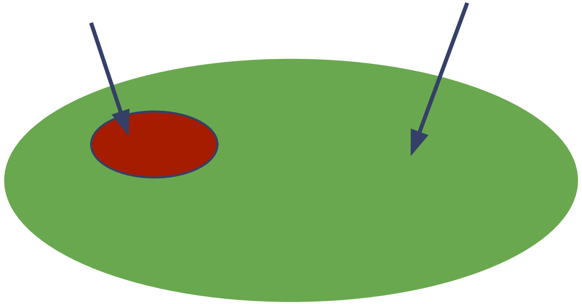
Tensor network:

Quantum many body wavefunction:

$$|\psi\rangle = \sum_{\{i_1,\ldots,i_n\}} C_{i_1,\ldots,i_n} |i_1, \ldots, i_n\rangle$$

$$\sim d^N$$





Computational cost:

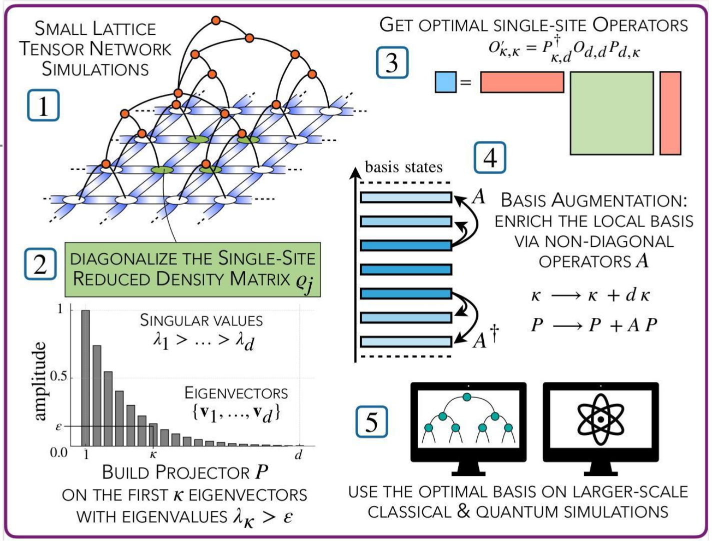
$$\mathcal{O}(d \cdot L \cdot BD^3) \qquad \mathcal{O}(d^2 \cdot L \cdot BD^3)$$

single - site tensor update

two - site tensor update

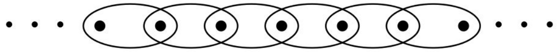
Optimal local basis truncation of lattice quantum many-body systems

Method:



- Estimate reduced density ρ_i

→ MF-theory, TN, Semi-definit programming

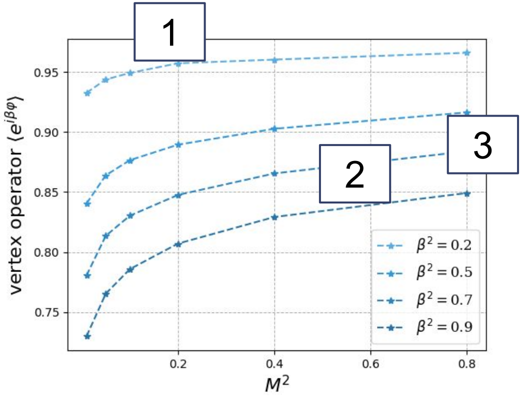


- Define projection P_i

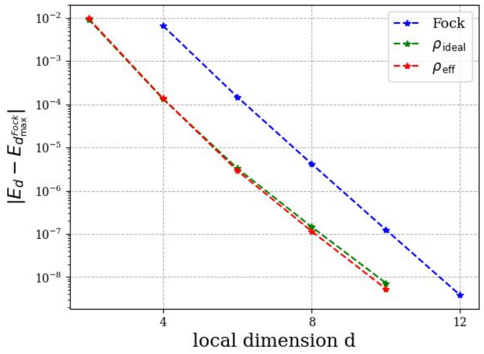
- Effective Hamiltonian $O_i^{\text{eff}} = P^\dagger O_i P$

sG-model:

$$aH = \sum_{j\in\Lambda} \pi_j^2 + \frac{1}{2}\left[(\varphi_{i+1}-\varphi_j)^2 - \frac{M^2}{\beta^2}\cos(\beta\varphi_j)\right]$$



1



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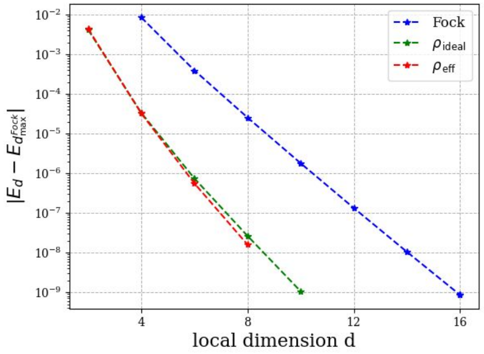

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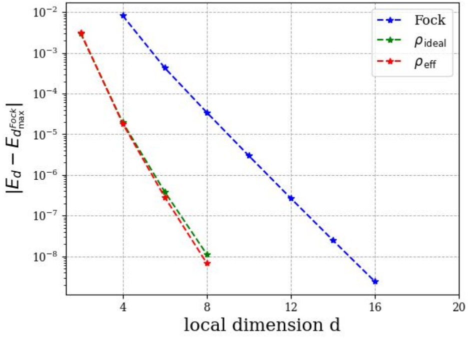
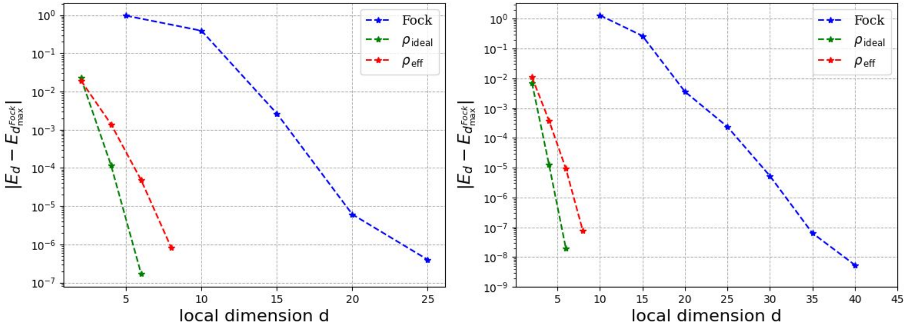


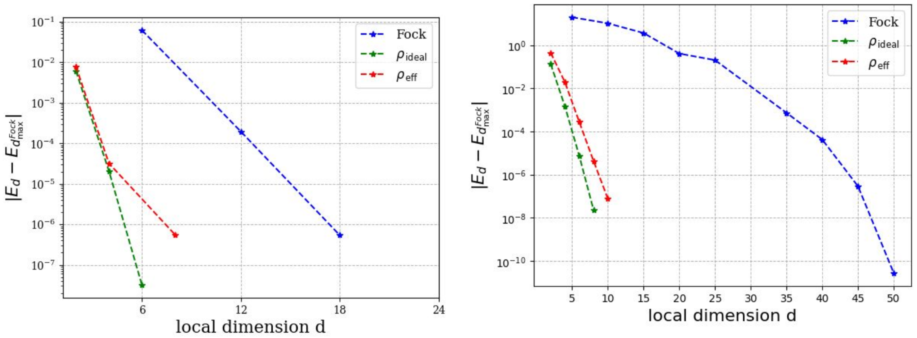
φ^4-model:

$$aH = \sum_{j \in \Lambda} \left[\frac{1}{2}\pi_j^2 + \frac{(\varphi_{j+1} - \varphi_j)^2}{2} + \frac{a^2 m^2}{2}\varphi_j^2 + \frac{a^2 \lambda}{4!}\varphi_j^4\right]$$

1D:



2D:



Lattice gauge theory dressed sites

Credits: Marco Rigobello

Hamiltonian LGT:

$$\mathrm{H} = \frac{i}{2}\sum_{x,k}\psi_x U_{x,k}\psi^\dagger_{x+k} + \mathrm{h.c} + m\sum_x(-1)^x\psi^\dagger_x\psi_x + \frac{g^2}{2}\sum_{x,k}E^2_{x,k} + \frac{1}{g^2}\sum_\Box P$$

Gauss law :

$$G^a_x\,|\psi_{\mathrm{phys}}\rangle = \left[\sum_{k=1}^{D}\left(L^a_{x,k} - R^a_{x,k}\right) - Q^a_x\right]|\psi_{\mathrm{phys}}\rangle = 0 \quad \forall x, a$$

Encoding for TN:

- Discard un-physical states: $\mathcal{H} = \otimes_x \mathcal{H}_x$

$$\boxed{[G_x, G_y] = 0}$$

- To preserve the tensor product structure we “double” symmetry content :

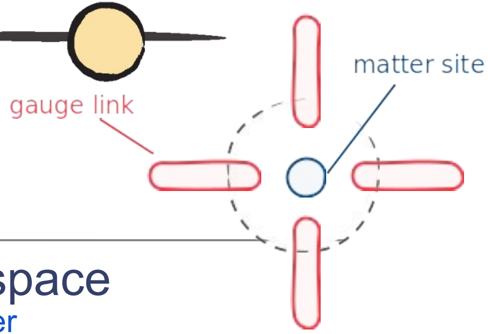
$$|j, m_L, m_R\rangle \longrightarrow |j, m_L\rangle \otimes |j, m_R\rangle$$

(in truncated irrep-basis)

split in semi-links

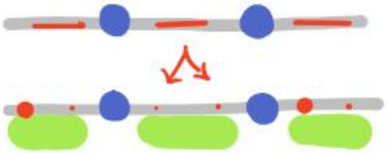
- Fuse site and surrounding sem-links and find singlets

- Impose abelian constraint on semi-links

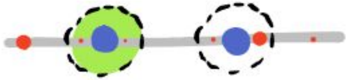


Algorithm

❶ split links



❷ combine site + semi-links



❸ select colour singlets (**Gauss law**)

local Hilbert space

fermionic matter

single component

$\{\psi_x, \psi_y^\dagger\} = \delta_{xy}$

bosonic gauge link

$|g\rangle \ \forall\, g \in \mathbf{SU}(3)$

infinite dimensional

Peter-Weyl $\mathscr{H} = \bigoplus_j (j \otimes j^\star)$

$|j, m, \bar{m}\rangle \ \forall$ irrep $j,\ m \in j,\ \bar{m} \in j^\star$

algebra $[E^a, U^\dagger] = iT^a U^\dagger$

$U\,|j, m, \bar{m}\rangle =$ Clebsch-Gordan

$E^2\,|j, m, \bar{m}\rangle = C_2(j)$ **Casimir**

Input:
(1+d)D
background charge
matter content
gauge group
truncation scheme

What gauge groups?

$$Z_{\mathrm{N}}, U(1), SU(N)$$

Non-simple groups:

$$U(1) \otimes SU(2) \otimes SU(3)$$

```
links: 2
gauge:
  G:
    group: SU 3
    num-C2-levels: 1
matter:
  u:
    irreps:
      G: 1 0 0
  d:
    irreps:
      G: 1 0 0
```

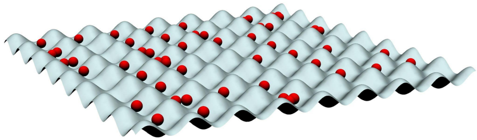
DressGauss

Output:
Operators

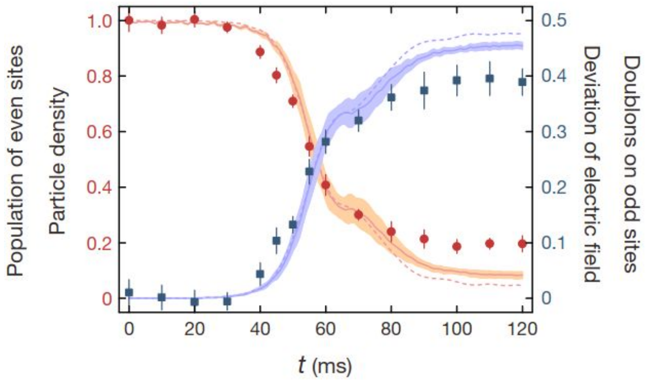
(gauge invariant)
(abelian constraint)
(local interaction)

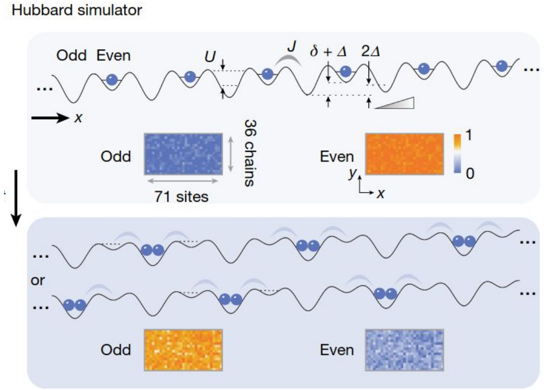
It’s modular! You can add your favourite gauge group or truncation.

Ultra-cold Atom Simulator for Quantum Link Model's in (2+1)D



What has been done?





Target model:

$$\hat{H}_{\mathrm{QLM}} = \sum_{\mathbf{r}} \left(-\frac{it}{2} \psi^\dagger_{\mathbf{r}} \hat{S}_{\mathbf{r},\mathbf{r}+1} \psi_{\mathbf{r}+1} + \mathrm{H.c} + m \sum_{\mathbf{r}} \psi^\dagger_{\mathbf{r}} \psi_{\mathbf{r}} \right)$$

$$\hat{G}_{\mathbf{r}} = (-1)^{\mathbf{r}+1} \left(\hat{S}^z_{\mathbf{r},\mathbf{r}+1} + \hat{S}^z_{\mathbf{r}-1,\mathbf{r}} + \psi^\dagger_{\mathbf{r}} \psi \right)$$

$\longleftrightarrow$

Simulator:

$$\hat{H}_{\mathrm{BHM}} = \sum_j \left[-J \left(\hat{b}^\dagger_j \hat{b}_{j+1} + \mathrm{H.c} \right) + \frac{U}{2} \hat{n}_j (\hat{n}_j - 1) + \left((-1)^j + j \vartriangle \right) \hat{n}_j \right]$$

Observation of gauge invariance in a 71-site Bose–Hubbard quantum simulator, Yang et al.Nature volume 587 (2020)

Spin S=1 U(1) Hamiltonian (2+1)D:

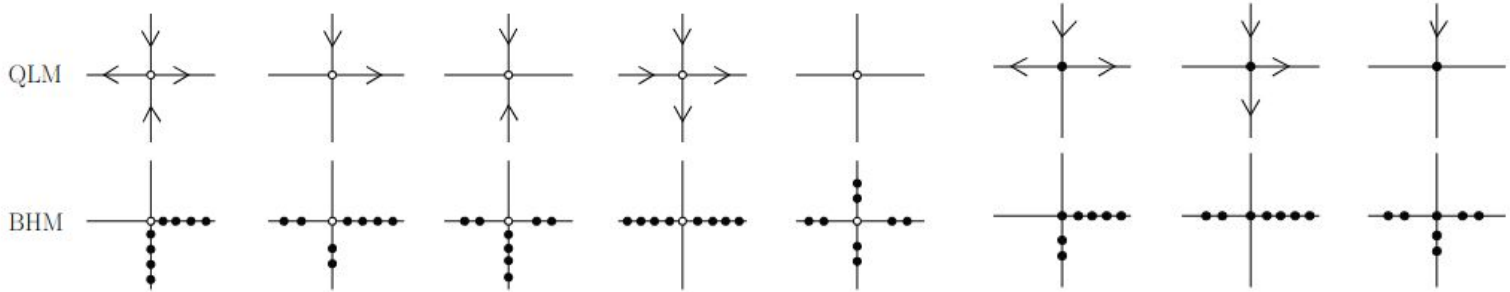
$$\hat{H}_{\text{QLM}} = \frac{\kappa}{2\sqrt{S(S+1)}} \sum_{\mathbf{r},\nu} \left(\psi^{\dagger}_{\mathbf{r}} \hat{S}_{\mathbf{r},\mathbf{r}+1} \psi_{\mathbf{r}+1} + \text{H.c} \right)$$

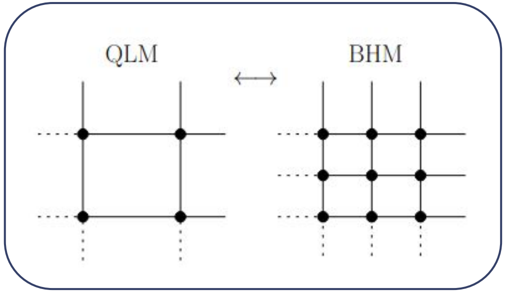
$$+ m \sum_{\mathbf{r}} \hat{\sigma}^{z}_{\mathbf{r}} + \frac{g^2}{2} \sum_{\mathbf{r},\nu} (\hat{s}^{z}_{\mathbf{r},e_\nu})^2$$

- Gauss operator:

$$\hat{G}_{\mathbf{x}} |\phi\rangle = |\phi\rangle \quad \forall \mathbf{x} \in \Lambda$$

Mapping:

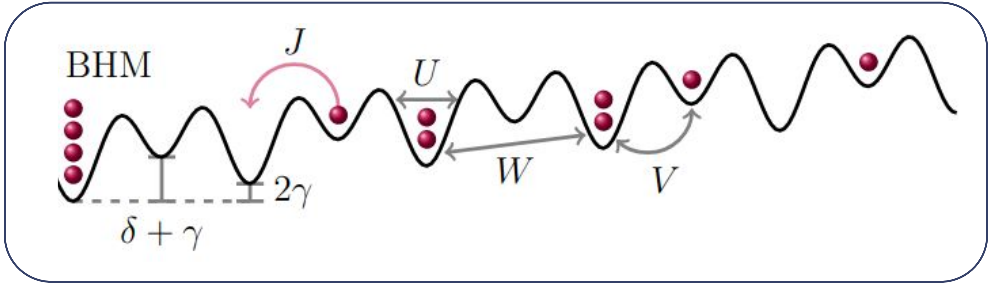




Ultracold-atoms setup:

$$\hat{H}_{\mathrm{BHM}} = \sum_{\mathbf{j},\nu} -J\left(\hat{b}^{\dagger}_{\mathbf{j}}\hat{b}_{\mathbf{j}+e_\nu} + \mathrm{H.c}\right) + \sum_{\mathbf{j}} \frac{U_{\mathbf{j}}}{2}\hat{n}_{\mathbf{j}}(\hat{n}_{\mathbf{j}} - 1) + \sum_{\mathbf{j}} \left(\gamma^T \mathbf{j} - \delta_{\mathbf{j}} - \eta_{\mathbf{j}}\right)\hat{n}_{\mathbf{j}}$$

$$+ V\sum_{\mathbf{j},\nu} \hat{n}_{\mathbf{j}}\hat{n}_{\mathbf{j}+e_\nu} + W \sum_{\mathbf{j},\nu,\nu''} \hat{n}_{2\mathbf{j}-e_\nu}\hat{n}_{2\mathbf{j}+e_{\nu'}}$$



Spin-S U(1) Quantum Link Models with Dynamical Matter on a Quantum Simulator, Jesse J Osborne et al.

Where is the gauge invariance?

- Gauge operator in BHM:

$$\hat{G}_{\mathbf{r}} = (-1)^{r_x + r_y} \left(\hat{n}_{\mathbf{r}} + \frac{1}{2} \sum_{\nu} \hat{n}_{\mathbf{r},\mathbf{e}_\nu} + \hat{n}_{\mathbf{r}-\mathbf{e}_\nu,\mathbf{e}_\nu} - 4 \right) .$$

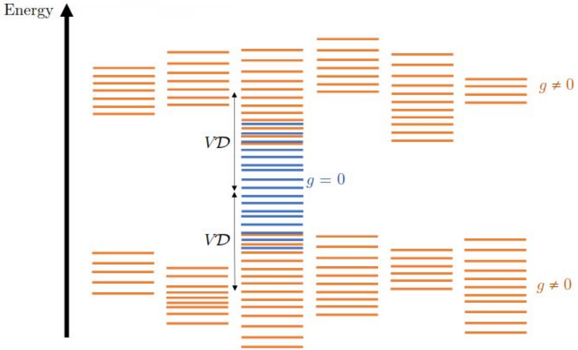
linear gauge protection

- We rewrite the BHM- Hamiltonian:

$$H_{\mathrm{BHM}} \supseteq \sum_{\mathrm{r}} -\delta \sum_{\nu} \hat{n}_{\mathrm{r}+\mathrm{e}_\nu} + 2\,(-1)^{r_x+r_y} \boxed{\vec{\gamma}^T \mathrm{r}\, \hat{G}_{\mathrm{r}}}$$

- In general:

$$H = H_0 + \lambda H_1 + V \sum_i c_i G$$

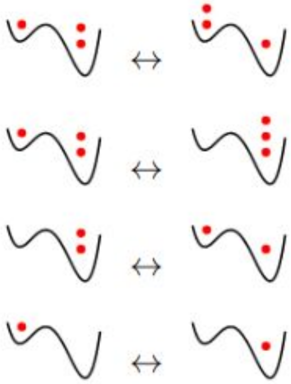


Gauge-Symmetry Protection Using Single-Body Terms, Jad C. Halimeh et al. PRX Quantum 2, 040311

Effective Hamiltonian:

$$(H_{\text{eff}})_{ij} = \delta_{ij} E_i + \frac{1}{2} \sum_k \left(\frac{1}{E_i - E_k} + \frac{1}{E_j - E_k} \right) V_{ik} V_{kj}$$

- Processes in the Bose–Hubbard model which do not correspond to any process in the QLM:



$$\sum_{\pm\gamma_i} \frac{16J^2}{(1-\alpha)U - 3V + 8W - \delta + \gamma_i}$$

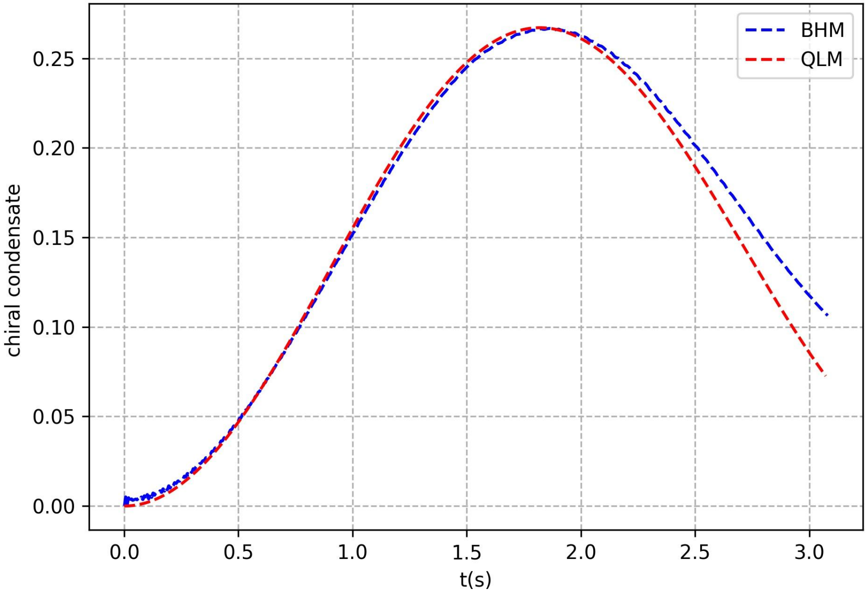
$$\sum_{\pm\gamma_i} \frac{12J^2}{-2U + 5V - 8W + \delta + \gamma_i}$$

$$\sum_{\pm\gamma_i} \frac{8J^2}{U - 7V + 12W - \delta + \gamma_i}$$

$$\sum_{\pm\gamma_i} \frac{2J^2}{5V - 12W + \delta + \gamma_i}$$

Quench dynamics:

$\mu/\kappa \approx -0.829$ and $g^2/\kappa = -0.0007$



Collaborators:

Marco Rigobello
Giovanni Cataldi
Jesse J. Osborne
Mattia Morgavi
Pietro Silvi
Jad C. Halimeh
Simone Montangero

Thank you!

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Backup

Half-link decomposition:

- The change of basis from group to representation basis is given by

$$\langle g|jmn\rangle = \sqrt{\frac{\dim(j)}{\mathrm{Vol}(G)}}[\pi_j(g)]_{mn}$$

- Naturally the parallel transporter can then be decomposed as:

$$\langle j'm'n'|U^J_{M,N}|jmn\rangle = \sqrt{\frac{\dim(j)}{\dim(j')}}\sum_\alpha \quad \overline{C^{j'n',\alpha}_{jmJM}}C^{j'm',\alpha}_{jnJN}$$